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# Generalized Bloch spheres for *m*-qubit states

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**Abstract**

$m$-qubit states are embedded in $\mathfrak{Cl}_{2m}$ Clifford algebras. Their probability spectrum then depends on $O(2m)$- or $O(2m+1)$-invariants, respectively. Parameter domains for $O(2m(+1))$-vector and -tensor configurations, generalizing the notion of a Bloch sphere, are derived.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

For many purposes it is useful to consider $m$-qubit states as vectors in an $\mathbb{R}$-linear Hilbert space $\mathfrak{H}$ whose basis is a set $\{B_i, i = 1, \dots, 2^{2m}\}$ of $2^m \times 2^m$ orthonormal

$$\mathrm{trace}(B_i \cdot B_j) = \delta_{ij},$$

Hermitian matrices:

$$\mathfrak{H} = \left\{ \sum_{k=1}^{2^{2m}} b_k B_k \mid b_k \text{ real} \right\} \qquad \text{(H)}.$$

A state is either represented by a Hermitian, normalized matrix or an appropriate coordinate vector $[b_1, b_2, \dots, b_{2^{2m}}]$ (a formulation in an appropriate *projective* space would be more adequate). In [2, 3] the generators of the quantum invariance group $SU(2^m)$ are proposed as such a basis, a possibility which we shall discuss in the summary.

A straightforward solution for the parametrization of a state $\varrho$ (a density matrix) is to write the set of all states as

$$\{\varrho\} = \bigcup_{\Lambda} \rho_\Lambda^{\mathfrak{U}},$$

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where

$$\rho_\Lambda^{\mathfrak{U}} = \{U^+ \varrho_\Lambda U \mid U \in \mathfrak{U}(2^m)\}$$

and

$$\varrho_\Lambda \text{ is the diagonal matrix } \varrho_\Lambda = \mathrm{diag}\{\Lambda\},$$
$$\Lambda = \left\{[\lambda_1, \lambda_2, \ldots, \lambda_{2^m}] \mid \lambda_i \text{ real}, \sum \lambda_i = 1, \lambda_i \geqslant 0\right\},$$

$\mathfrak{U}(2^m)$ is the unitary group in $2^m$ dimensions and $\Lambda$ is the probability spectrum generating the state $\rho_\Lambda^{\mathfrak{U}}$. This construction warrants positivity and normalization. It is however not always (or, better, almost never) convenient in the discussion of physical situations[2]. On the other hand writing $\varrho$ as a vector in $\mathfrak{H}$ confronts us with the problem of deriving conditions for the expansion coefficients[3] that guarantee the expansion to yield a state. Formulated in this general way the problem has no obvious solution: positivity and normalization conditions can be derived by expressing the eigenvalues in terms of the expansion coefficients, i.e. finding the zeros of the characteristic polynomial as functions of these parameters. As we know from Abel and Galois a solution by rational operations and radicals does not exist for quintic or higher degrees, i.e. for general 3-qubit and *a fortiori* for higher systems. For the 2-qubit explicit expressions are given by the Ferrari–Cardano formulae.

In this paper I explicitly construct classes of states for all $m$ whose spectra are determined by characteristic polynomials factorizing into polynomials of a given degree. The novel point in our considerations is the use of *Hermitian matrix representations of a Clifford algebra* to construct bases in $\mathfrak{H}$. This particular choice of basis allows us to arrange the $2^{2m} - 1$ real coordinates of an $m$-qubit state in multidimensional arrays which are shown to 'transform' as $O(2m)$ tensors. This fact implies that the probability spectrum of an $m$-qubit state depends only on $O(2m)$-invariants, a considerable simplification of the parameter dependencies of these eigenvalues, indeed. This simplification leads to a complete characterization of complete[4] sets of states which allow for an explicit construction of a parameter domain. In this way I find the set of all states (vector states) whose parameter domain is the Bloch $2m$-sphere. Furthermore a set of (bivector) states is proposed whose novel parameter domain generalizes the notion of a Bloch sphere. Beyond these two domains the Descartes rule for the positivity of polynomial roots can be used to derive admissible parameter domains.

## 2. *m*-qubit states imbedded in Clifford algebras

An $m$-qubit system is controlled by $m$ spin degrees of freedom and hence by $2^{2m} - 1$ parameters (see footnote 2 on page 2). The determining anticommutation relation for Clifford numbers [1] ($\mathbb{I}$ is the unity)

$$\Gamma_i \cdot \Gamma_j + \Gamma_j \cdot \Gamma_i = 2\delta_{ij}\mathbb{I} \tag{1}$$

with

$$i, j = 1, \ldots, 2m$$

has $2^m$-dimensional, Hermitian, traceless matrix representations $\Gamma_j^{\{m\}}$.

[2] This is equally true for the parametrization $\varrho = \mathrm{e}^A/\mathrm{trace}(\mathrm{e}^A)$, $A$ Hermitian, which is rather clumsy e.g. when it comes to the discussion of separability conditions.

[3] These parameters are linearly related, a matrix representation of the basis in $\mathfrak{H}\, B_i, i = 1, \ldots, 2^{2m}$ given, to the matrix elements of the density matrix.

[4] Complete in the sense that all states factorizing in a specific way are contained in this set.

From the anticommutation relations we see immediately that the products

$$\Gamma_{j_1,j_2,\ldots,j_k} := i^{k-1}\Gamma_{j_1}\cdot\Gamma_{j_2}\cdots\Gamma_{j_k}, \tag{2}$$

$$k = 2,\ldots,m, \tag{3}$$

are totally anti-symmetric in the indices $[j_1,\ldots,j_k]$. The only symmetric object constructed from Clifford numbers is the unity

$$\mathbb{I} = \Gamma_i^2$$

as we see from the anticommutation relations. A product consists of at most $2m$ factors. Hence we have

$$\sum_{k=0}^{2m}\binom{2m}{k} = 2^{2m}$$

independent products. Furthermore because of the commutation relations we have

$$\mathrm{trace}\Big(\big(\Gamma_{i_1}^{\{m\}}\cdot\Gamma_{j_1}^{\{m\}}\cdots\Gamma_{k_1}^{\{m\}}\big)^{+}\Gamma_{i_2}^{\{m\}}\cdot\Gamma_{j_2}^{\{m\}}\cdots\Gamma_{k_2}^{\{m\}}\Big) \sim \sum\big(\delta_{\bar{i}_1\bar{i}_2}\delta_{\bar{j}_1\bar{j}_2}\ldots\delta_{\bar{k}_1\bar{k}_2}\big),$$

where the $\delta$-function expresses pairwise equality of the $\cdot_1$- and $\cdot_2$-indices.

A Hermitian $2^m\times 2^m$-matrix requires $2^{2m}$ real numbers for a complete parametrization. Thus $m$-qubit states can be expanded in terms of $\mathbb{I}$ and the products introduced: Clifford numbers are the starting point for the construction of a basis in the $\mathbb{R}$-linear space of Hermitian matrices: this basis is construed as a Clifford algebra $\mathfrak{Cl}_{2m}$ ($2^{2m}$-dimensional as we have seen). The important advantage to gain from this choice of basis is that now domains for parameters are determined by $O(2m)$-invariants. The number of parameters necessary for the specification of these domains is thus considerably reduced. For the domains found in this paper, this means one invariant for the vector-state configuration ($2m$ parameters) and two invariants for the bivector states ($m(2m-1)$ parameters) to be constructed below for all $m$.

I should remark that many beautiful geometric reverberations of Clifford algebras will play no role here, only very elementary properties of Clifford algebras will be sketched, emphasizing practical aspects. It is in this sense that the following, hopefully self-contained, outline of the method should be understood.

To construct a basis and its matrix representation $\mathfrak{G}^{\{m\}}$ in $\mathfrak{H}$, I proceed as follows:

- The product

$$\Gamma_{2m+1}^{\{m\}} := (-i)^m\Gamma_1^{\{m\}}\cdot\Gamma_2^{\{m\}}\cdots\Gamma_{2m}^{\{m\}} \tag{4}$$

  obviously anti-commutes with all the $\Gamma_i^{\{m\}}$, $i = 1,\ldots,2m$.
- The explicitly anti-symmetric products ($\varepsilon$ is the totally anti-symmetric symbol in $2m$ dimensions)

$$\begin{aligned}\hat{\Gamma}_{i_1,\ldots,i_k}^{\{m,k\}} &= F_{\mathrm{Norm}}^{\{m,k\}}\big(\varepsilon_{i_1,\ldots,i_{2m}}\Gamma_{i_{k+1}}^{\{m\}}\cdots\Gamma_{2m}^{\{m\}}\big)\Gamma_{2m+1}^{\{m\}}\\ F_{\mathrm{Norm}}^{\{m,k\}} &= \frac{(-i)^{m+s}}{(2m)!}\binom{2m}{k}\end{aligned} \tag{5}$$

$$s = \begin{cases}0 & \text{when } x = 0, 1\\ 1 & \text{when } x = 2, 3\end{cases} \quad \text{where } x = k \bmod (4). \tag{6}$$

The limiting cases $k = 1$ and $k = 2m$ are immediately seen to be

$$\hat{\Gamma}^{\{m,1\}}_{i_1} = (-1)^{m+1}\Gamma^{\{m\}}_{i_1}$$

$$\hat{\Gamma}^{\{m,2m\}}_{i_1,\dots,i_{2m}} = \frac{f_m}{2m!}\varepsilon_{i_1,\dots,i_{2m}}\Gamma^{\{m\}}_{2m+1}$$

with

$$f_m = \begin{cases} (-1)^{\frac{m}{2}} & \text{for } m \text{ even} \\ (-1)^{\frac{m+1}{2}} & \text{for } m \text{ odd.} \end{cases}$$

- Because of the anti-commutation relations the only symmetric tensor is the scalar, i.e. the unit matrix

$$\hat{\Gamma}^{\{m,0\}} = \mathbb{I}. \tag{7}$$

- The set of matrices $\mathfrak{G}^{\{m\}} = \{\hat{\Gamma}^{\{m,0\}}, \hat{\Gamma}^{\{m,1\}}, \dots, \hat{\Gamma}^{\{m,2m\}}\}$ is orthonormal in the sense of (1).
- Formally, this gives an identification of the linear spaces $\mathfrak{g}^{\{m,k\}} = \operatorname{span}(\hat{\Gamma}^{\{m,k\}}, \mathbb{R})$ and the tensor algebra $\bigwedge^k \mathbb{R}^{2m}$ of $\mathbb{R}^{2m}$. In detail we write[5]

$$\begin{array}{rcl} & \text{isomorphic vector spaces:} & \\ \text{scalar, } \mathbb{R} & | & \mathfrak{g}^{\{m,0\}} = \mathbb{R}\cdot\mathbb{I} \\ \text{vector, } \bigwedge^{1}\mathbb{R}^{2m} = \mathbb{R}^{2m} & | & \hat{\mathfrak{g}}^{\{m,1\}} \\ \text{(2-)tensor (bivector), } \bigwedge^{2}\mathbb{R}^{2m} & | & \hat{\mathfrak{g}}^{\{m,2\}} \\ & \cdots\cdots & \\ \text{volume element, } \bigwedge^{2m}\mathbb{R}^{2m} & | & \hat{\mathfrak{g}}^{\{m,2m\}}. \end{array}$$

- Following these observations we organize the state parameters in terms of a scalar $G_0^{\{m,0\}}$ and the totally anti-symmetric real arrays

$$G^{\{m,1\}}_{i_1}, G^{\{m,2\}}_{i_1,i_2}, \dots, G^{\{m,2m\}}_{i_1,i_2,\dots,i_{2m}}$$

(i.e. totally antisymmetric arrays of real numbers)

and thus account for

$$\sum_{k=0}^{2m}\binom{2m}{k} = 2^{2m} \tag{8}$$

coefficients.
- We write the expansion of an $m$-qubit state as

$$\varrho^{\{m\}} = \sum_{k=0}^{2m} G^{\{m,k\}} \circ \hat{\Gamma}^{\{m,k\}}, \tag{9}$$

where $\circ$ indicates the contraction $A \circ B = \sum_{i_1,\dots,i_k} A_{i_1,\dots,i_k} B_{i_1,\dots,i_k}$.
- An explicit construction of the representation $\Gamma^{\{m\}}_1, \dots, \Gamma^{\{m\}}_{2m}$ traditionally proceeds e.g. as follows:
starting with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we have the iteration scheme

$$\mathfrak{G}^{\{m+1\}} = \{\Gamma^{\{m,1\}} \times \sigma_1, \dots, \Gamma^{\{m,2m\}} \times \sigma_1, \Gamma^{\{m,0\}} \times \sigma_2, \Gamma^{\{m,0\}} \times \sigma_3\}. \tag{10}$$

[5] We use the slightly old fashioned notation: vector, tensor, ... $k$-tensor instead of vector, bivector, ... $k$-vector.

- $O(2m)$-symmetry. To begin with it might be useful to remind the reader the machinery of rotations in classical systems. Consider a canonical, classical system with $2m$ degrees of freedom, i.e. with a $2m$-dimensional configuration space. Infinitesimal $2m$-dimensional rotations and translations generated by generators

$$J_{i,j},\ P_i \quad \text{respectively}$$

($\{A, B\}$ denote Poisson brackets for functions defined on the phase space of the system) are defined as

Infinitesimally,
$F \longrightarrow F + \epsilon_1 \alpha_{i,j} \{J_{i,j}, F\} + \epsilon_2 \beta_i \{P_i, F\}$
(repeated indices are summed over)
where
$\epsilon$ is infinitesimal and $\alpha_{i,j}$, $i, j = 1 \ldots 2m$,
is an antisymmetric array of parameters
and the $\beta_i$ parametrize translations.

The Lie algebra of the Euclidean–Poincaré group is

$$\{J_{i,j}, J_{k,l}\} = \delta_{i,l} J_{j,k} + \delta_{j,k} J_{i,l} - \delta_{i,k} J_{j,l} - \delta_{j,l} J_{i,k}$$
$$\{J_{i,j}, P_k\} = P_i \delta_{j,k} - P_j \delta_{i,k}.$$

The anticommutation relations (1) defining the Clifford algebra $\mathfrak{Cl}_{2m}$ spanned by the set of totally antisymmetric products and the unity, $\mathfrak{G} = \{\mathbb{I}, \Gamma_i, i\Gamma_i\Gamma_j, \ldots\}$ considered above, lead to an analogous algebraic structure. A straightforward calculation shows ($\Gamma_{i,j} := \mathrm{i}\Gamma_i \cdot \Gamma_j$)

$$\frac{\mathrm{i}}{2}[\Gamma_{i,j}, \Gamma_{k,l}] = \delta_{i,l}\Gamma_{j,k} + \delta_{k,j}\Gamma_{i,l} - \delta_{l,j}\Gamma_{i,k} - \delta_{i,k}\Gamma_{j,l} \tag{11}$$
$$\frac{\mathrm{i}}{2}[\Gamma_{i,j}, \Gamma_k] = \delta_{k,i}\Gamma_j - \delta_{j,k}\Gamma_i. \tag{12}$$

These relations constitute a quantum analogue of the classical representation of the $O(2m)$ Lie algebra[6]: the $\Gamma_{i,j}$ generate rotations, the $\Gamma_i$ translations, the array $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_{2m}\}$ 'transforms as a vector'. The basis elements of the dual Grassmann algebra $\bigwedge \mathbb{R}^{2^m}$ can be identified with (see above).
$\mathbb{G} = \{G^{\{m,0\}}, G^{\{m,1\}}, \ldots, G^{\{m,2m\}}\}$ and 'transform as tensors'. More precisely we have

$$L \in O(2m) \longmapsto U(L) = \mathrm{e}^{-\frac{\mathrm{i}}{4}\alpha_{i,j}\Gamma_{i,j}} \tag{13}$$

$O(2m)$-transformations

$$G_i^{\{m,1\}} \longmapsto L_{i,k} G_k^{\{m,1\}} \tag{14}$$
$$G_{i,j}^{\{m,2\}} \longmapsto L_{i,i_1} L_{j,j_1} G_{i_1,j_1}^{\{m,2\}} \tag{15}$$

etc induce transformations

$$\Gamma_i \longmapsto U(L)\Gamma_i U(L)^{-1} = (L^{-1})_{i,k}\Gamma_k \tag{16}$$
$$\Gamma_i\Gamma_j \longmapsto U(L)\Gamma_i\Gamma_j U(L)^{-1} = (L^{-1})_{i,i_1}(L^{-1})_{k,k_1}\Gamma_{i_1}\Gamma_{k_1} \tag{17}$$
$$\cdots.$$

[6] Precisions concerning a more precise discussion of the universal covering group are of no avail here and will not be touched.

Configurations parametrized by one of the tensors $G^{\{m,k\}}$ have some comfortable (and profitable) properties. For instance the coefficients of the characteristic polynomials are expected to depend on $O(2m)$-invariants built from these tensors. Furthermore the probability spectra will exhibit degeneracy patterns corresponding to the rank of the tensors $G^{\{m,k\}}$; parameter ranges corresponding to physical states will be determined by universal polynomials in terms of these invariants.

The following sections are devoted to detailed discussions of these observations for the cases of $m = 2$, 3-qubits. General results for $m$-qubits will be derived.

## 3. $O(2^m)$-tensor configurations

In this chapter I introduce some nomenclature which derives from similar objects occurring in the Dirac theory of relativistic Fermions.

The iteration scheme (10) provides us with explicit bases for Clifford algebras $\mathfrak{Cl}_{2m}$.

The coordinates representing an $m$-qubit introduced in equation (H) of the introduction are organized in

- scalar $G^{\{m,0\}}$, $G^{\{m,0\}} = 1$ because of state normalization
- vector $G^{\{m,1\}}$,
- 2,3-tensor $G^{\{m,2,3\}}$, and
- pseudoscalar $G^{\{m,2m\}}$,
- pseudovector $G^{\{m,2m-1\}}$,
- pseudotensor $G^{\{m,2m-(2,3)\}}$

components[7].

- $m = 1$. The 2-Clifford algebra is spanned by[8]

$$\begin{aligned} \hat{\Gamma}^{\{1,0\}} &= \sigma_4 && \text{scalar} \\ \hat{\Gamma}^{\{1,1\}} &= \{\sigma_1, \sigma_2\} && \text{vector} \\ \hat{\Gamma}^{\{1,2\}} &= \sigma_3 && \text{pseudoscalar.} \end{aligned} \tag{19}$$

A qubit state is then written as ($G^{\{m,o\}} = \frac{1}{2^m}$ because of normalization)

$$\varrho = \tfrac{1}{2}(G^{\{1,0\}}\hat{\Gamma}^{\{1,0\}} + G^{\{1,1\}} \circ \hat{\Gamma}^{\{1,1\}} + G^{\{1,2\}}\hat{\Gamma}^{\{1,2\}}) \tag{20}$$

- $m = 2$. The Clifford algebra is now spanned by

$$\begin{aligned} &\hat{\Gamma}_1^{\{2,1\}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \hat{\Gamma}_2^{\{2,1\}} = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix} \\ &\hat{\Gamma}_3^{\{2,1\}} = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix} \qquad \hat{\Gamma}_4^{\{2,1\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned} \tag{21}$$

[7] Here we follow the nomenclature of Dirac theory (generalized for $m \neq 2$) for relativistic fermions choosing an Euclidean–Majorana representation for $\hat{\Gamma}^{\{m,k\}}$ generated by the iteration scheme (10).

[8] We could have chosen

$$\hat{\Gamma}^{\{1,0\}} = \sigma_4 \qquad \hat{\Gamma}^{\{1,1\}} = \{\sigma_2, \sigma_3\}, \text{ or } \{\sigma_3, \sigma_1\} \qquad \hat{\Gamma}^{\{1,2\}} = \sigma_1, \text{ or } \sigma_2 \tag{18}$$

as well. Both basis are connected by an $O(2)$ rotation by $\pi/4$.

We write

$$\varrho = \tfrac{1}{4}(G^{\{2,0\}}\hat{\Gamma}^{\{2,0\}} + G^{\{2,1\}} \circ \hat{\Gamma}^{\{2,1\}} + G^{\{2,2\}} \circ \hat{\Gamma}^{\{2,2\}} + G^{\{2,3\}} \circ \hat{\Gamma}^{\{2,3\}} + G^{\{2,4\}}\hat{\Gamma}^{\{2,4\}}). \quad (22)$$

The iteration algorithm (10) straightforwardly provides analogous representations for $m > 2$.

## 4. Probability spectra for tensor configurations and their degeneracies

In this section we explicitly determine the $m = 1, 2, 3$ probability spectra of the vector and tensor configurations by calculating the roots of the characteristic polynomial

$$P^{\{m\}} := \text{determinant } (\varrho^{\{m\}} - \lambda\mathbb{I}).$$

Parameter domains generalizing Bloch spheres are obtained by requiring that the spectrum obtained be a probability distribution. Degeneracies of $m$-qubit tensor spectra are shown to follow simple patterns. Because of the normalization condition a 'tensor configuration' always reads as

$$\varrho_{k_{\text{tensor}}} = \frac{1}{2^m}(\mathbb{I}_m + G^{\{m,k_{\text{tensor}}\}} \circ \hat{\Gamma}^{\{m,k_{\text{tensor}}\}}). \quad (23)$$

We find

- *Vector configurations*: $k_{\text{tensor}} = 1$. The probability spectra are $2^{m-1}$-fold degenerate, i.e. built up by one doublet repeated $2^{m-1}$ times. The doublet is found to be

$$\lambda = \frac{1}{2^m}(1 \pm \|G^{\{m,1\}}\|), \quad (24)$$

where the absolute value of $G^{\{m,1\}}$ is simply the vector norm

$$\|G^{\{m,1\}}\| = \left(\sum_{i=1}^{2m} \left(G_i^{\{m,1\}}\right)^2\right)^{1/2}. \quad (25)$$

**Remarks**

– The inclusion of the pseudoscalar $G^{\{m,2m\}}$ leads to an additional dimension. We have

$$\lambda = \frac{1}{2^m}(1 \pm \|\tilde{G}\|), \quad (26)$$

where

$$\|\tilde{G}\| = \left(\sum_{i=1}^{2m} \left(G_i^{\{m,1\}}\right)^2 + (G^{\{m,2m\}})^2\right)^{1/2}. \quad (27)$$

– For pure states the parameter domains are, of course,

the $(2m-1)$-sphere $\|G^{\{m,1\}}\| = 1$ for vector configurations (28)

and the $2m$-sphere $\|\tilde{G}\| = 1$ for the pseudoscalar + vector configuration. (29)

– Mixed states are represented by the corresponding spheres with radius $\|\tilde{G}\| < 1$.

- *2-tensor configurations*: $k_{\text{tensor}} = 2$. Probability spectra turn out to be $2^{m-2}$-fold degenerate: a spectrum is built up by one quartet repeated $2^{m-2}$ times. We express these four eigenvalues in terms of $O(2m)$-invariants. In the following we shall present explicit calculations for the cases $m = 2$ and $m = 3$ and then generalize our findings to the general case.

– $m = 2$. The eigenvalues are

$$\lambda = \tfrac{1}{4}\big(1 \pm \sqrt{r \pm \sqrt{2r^2 - T_4}}\big), \tag{30}$$

where

$$r = \tfrac{1}{2}\,\mathrm{trace}((G^{\{m,2\}})^T \cdot G^{\{m,2\}}) \quad \text{(Frobenius norm)}^2 \tag{31}$$

$$T_4 = \mathrm{trace}\big(\big(G^{\{m,2\}^T} \cdot G^{\{m,2\}}\big)^2\big). \tag{32}$$

We see that the eigenvalues depend on only two invariants $r$ and $T_4$.[9]

– $m = 3$. For $m \geqslant 3$ new invariants appear (see the discussion at the end of this section), the characteristic polynomial $P^8$ can be shown to factorize into 2 polynomials $P_{4,\pm}$ of degree 4 which differ by the sign of $D^{(3)}$.

$$\begin{aligned} P_{4,\pm}(z) = z^4 - z^3/2 + (3-r)z^2/32 + \left(r - 1 \pm \frac{64}{3} D^{(3)}\right) z/128 \\ + \left(2 - (r+1)^2 + T_4 \mp \frac{256}{3} D^{(3)}\right) \Big/ 4096, \end{aligned} \tag{33}$$

where

$$D^{(3)} = \epsilon_{i_1,i_2,i_3,i_4,i_5,i_6} G^{\{m,2\}}_{i_1,i_2} G^{\{m,2\}}_{i_3,i_4} G^{\{m,2\}}_{i_5,i_6} \tag{34}$$

(as usual repeated indices are summed over).
The eigenvalues of $P_{4,\pm}$ are even in $D^{(3)}$ and depend only on $(D^{(3)})^2$:
*the octet of eigenvalues therefore is degenerate in 2 quartets.*
Under the assumption that the 2-tensor configuration is such that $D^{(3)}$ vanishes we again find the $m = 2$ relation

$$\lambda = \tfrac{1}{8}\big(1 \pm \sqrt{r \pm \sqrt{2r^2 - T_4}}\big)$$

if $D^{(3)} = 0$

$r$ is the Frobenius norm and

$T_4$ is the trace invariant of scale dimension 4 defined above.

The degeneracy into 2 quartets is explicitly seen in this case.

– $m = m_0 \geqslant 4$. At this stage of affairs the following 'Vermutung' is plausible:

A $k_{\mathrm{tensor}} = k_0 \leqslant m_0$ configuration is $2^{m_0-k_0}$-fold degenerate and consists of $2^{m_0-k_0}$ $2^{k_0}$-plets. Algebraic solutions of the spectral decomposition can be found for $k_0 \leqslant 2$ and *all m*. A direct though not particularly elegant proof of this 'Vermutung' is possible by calculating the characteristic polynomial of the corresponding tensor configuration using e.g. the relation $\mathrm{Det}\, A = \mathrm{e}^{\mathrm{trace}(\log A)}$. For example it is easily seen that for vector configurations $k_{\mathrm{tensor}} = 1$ the characteristic polynomial $P^{\{m\}}$ factorizes as proposed for all $m$

$$P^{\{m\}} = \left(\lambda^2 - \frac{\lambda}{2^{(m-1)}} + \frac{(1-r)}{2^{2m}}\right)^{2^{m-1}}.$$

For 2-tensor configurations, we obtain the following:

[9] In the definition of the Frobenius norm we include, because of (anti-)symmetry, a factor of 1/2: $r = \sum_{i<j}^{2m} m_{ij}^2$, where $(m_{ij})$ is a $2m \times 2m$ (anti-)symmetric matrix.

defining

$$\bar{P}_{\pm} := z^4 - 4z^3 + 2(3-r)z^2 - \left(r - 1 \pm \frac{64}{3} D^{(3)}\right) z + \left(2 - (r+1)^2 \mp \frac{256}{3} D^{(3)} + T_4\right)$$

we find

$$P^{\{m\}}(\lambda) = (\bar{P}_+ \bar{P}_-)^{2^{m-3}}|_{z=2^m\lambda}.$$

I shall not spell out the not very inspiring details.

Anticipating the discussion proposed in the next paragraph we shall sketch a proof that the occurrence of a third order invariant $D^{(3)}$ is possible for $m \geqslant 4$. For $m = 3$ the $O(2m)$ 2-tensor allows for the construction of an anti-symmetric 2-tensor of scale dimension 2 given two 2-tensors

$$\tilde{A}_{i_1,i_2} = \epsilon_{i_1,i_2,i_3,i_4,i_5,i_6} G^{\{3,2\}}_{i_3,i_4} G^{\{3,2\}}_{i_5,i_6} \tag{35}$$

and therefore of the invariant

$$D^{(3)} = \mathrm{trace}(\tilde{A} \cdot G^{\{3,2\}}),$$

for $m \geqslant 4$ higher tensors $G$ are required to be contracted to third order invariants (i.e. scale dimension = 3 (see below)). We see that the roots of the 4th order polynomials—4th order because of the degeneracy of the 2-tensor configurations described above—are functions of the two even invariants $r$ and $T_4$ defined in (31) and (32) and a third order invariant. The eigenvalues are expected to be given by (30) for all $G^{\{m,2\}}$ under the condition $D^{(3)} = 0$ constructed in the way discussed.

– The parameter domains for 2-tensor configurations are now determined in a straightforward manner. In a $(r - T_4)$-diagram positivity and normalization leads to the inequality

$$\max((r+1)^2 - 2, 0) \leqslant T_4 \leqslant 2r^2, \qquad 0 \leqslant r \leqslant 1, \tag{36}$$

for the admissible $r, T_4$ values. Figure 1 shows the corresponding diagram.

It is more convenient to introduce new variables

$$(r, T_4) \longrightarrow \left(r, z = \tfrac{1}{2} - \sqrt{2r^2 - T_4}\right),$$

the inequalities (37) now read

$$\begin{array}{ll} \tfrac{1}{2} - z \leqslant r \leqslant \tfrac{1}{2} + z & \quad 0 \leqslant z \leqslant \tfrac{1}{2} \\ \tfrac{1}{2} + z \leqslant r \leqslant \tfrac{1}{2} - z & \quad -\tfrac{1}{2} \leqslant z \leqslant 0. \end{array}$$

In terms of $N_m := m(2m-1)$ matrix elements of $G^{\{m,2\}}$ the invariants $r, T_4$ or $r, z$ correspond to the following geometrical 'balls':

For all $m$ we have

$$r = \sum_{i<j}^{2m} \left(G^{\{m,2\}}_{i,j}\right)^2 \tag{37}$$

i.e. the $(N_m - 1)$-sphere of the Frobenius norm

For $m = 2$:

a straightforward calculation shows

$$2r^2 - T_4 = \frac{1}{16} \mathrm{trace}(\tilde{G}^{\{2,2\}} \cdot G^{\{2,2\}})^2, \tag{38}$$

where

$$\tilde{G}^{\{2,2\}} \text{ is the dual tensor } \tilde{G}^{\{2,2\}}_{i_1,i_2} = \epsilon_{i_1,i_2,i_3,i_4} G^{\{2,2\}}_{i_3,i_4}.$$

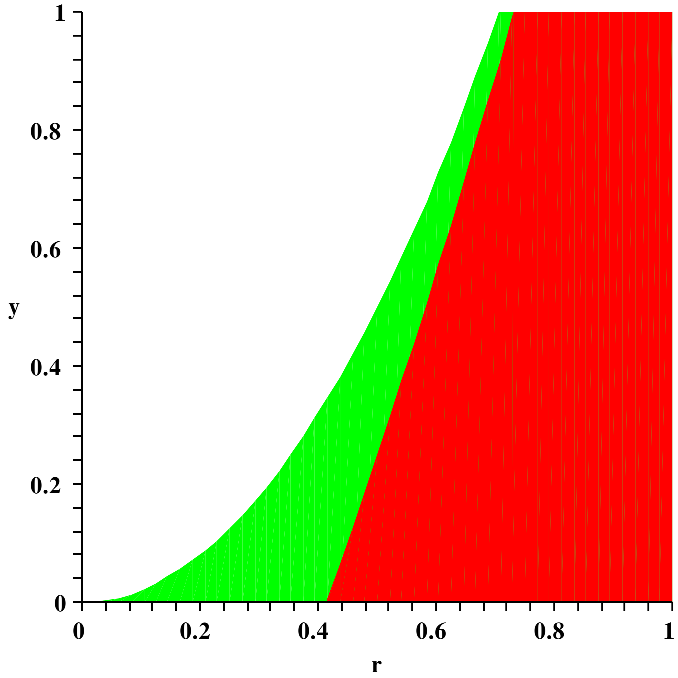


**Figure 1.** The admissible 2-tensor domain in a $(r, T_4)$-plot.

($\tilde{A} := \tilde{G}^{\{3,2\}}$ the tensor dual to $G^{\{3,2\}}$)

For $m = 3$ we find

$$2r^2 - T_4 = \tfrac{1}{16}\|\tilde{A}\|^2 = \tfrac{1}{32}\,\mathrm{trace}(\tilde{A}^T\tilde{A}) \tag{39}$$

$\|\tilde{A}\|$ is the Frobenius norm of the 2-tensor (32).

The parameters $a_{i,k} := G^{\{m,2\}}_{i,k}$ are seen to lie in generalized elliptic 'tunnel' domains (see below). In detail we have

– $m = 2$. Expressing $z$ in terms of $a_{i,k}$

$$z = \tfrac{1}{2} - 2|a_{1,2}a_{3,4} + a_{1,3}a_{4,2} + a_{1,4}a_{2,3}| \tag{40}$$

remember $\quad \lambda = \frac{1}{4}\left(1 \pm \sqrt{r \pm \left(\frac{1}{2} - z\right)}\right)$

we shall see that the admissible values $-\frac{1}{2} \leqslant z \leqslant \frac{1}{2}$ lie in a generalized elliptic 'tunnel' domain embedded in $\mathbb{R}^6$.

– $m = 3$. The analogous expression reads in this case

$$\begin{aligned} z = \tfrac{1}{2} - 2((a_{4,5}a_{3,6} - a_{4,6}a_{3,5} + a_{3,4}a_{5,6})^2 \\ + (a_{2,5}a_{4,6} - a_{4,5}a_{2,6} + a_{2,4}a_{3,6})^2 + 13 \text{ similar terms})^{1/2} \end{aligned} \tag{41}$$

and $z = z(a_{i,k})$ is the 'tunnel' domain embedded in $\mathbb{R}^{15}$ which we shall illustrate at the end of the section.

- I should include a short discussion of a qualitative method for the construction of $O(2m)$-invariants by inference. First of all we assign the scale dimension $\delta = 1$ to the tensor $G^{\{m,k\}}$ all $m, k$. The eigenvalues then have $\delta = 1$, the characteristic polynomial $P(z) = \sum_{i=0}^{d} c_i z^i$ of degree $d$ has $\delta = d$, the coefficients have $\delta^{d-i}$. Hence $c_i$ is composed of invariants of scale dimension $\leqslant d - i$ (counting dimensions such that by putting $G^{\{m,0\}} = 1$ (and thus fulfilling the normalization condition) we mean that unity carries one-dimensional unit). In detail we have the following invariants:

– $\delta = d = 4$ : $T_4$ *and* $r^2$. We reiterate the identities introduced above.
$m = 2$:

$$2r^2 - T_4 = \tfrac{1}{16}\left(\epsilon_{i_1,i_2,i_3,i_4} G^{\{2,2\}}_{i_1,i_2} G^{\{2,2\}}_{i_3,i_4}\right)^2 = 4 \text{ determinant } (G^{\{2,2\}}) \tag{42}$$

$m = 3$:

$$2r^2 - T_4 = \tfrac{1}{32}\epsilon_{i_1,i_2,i_3,i_4,i_6,i_5} G^{\{3,2\}}_{i_1,i_2} G^{\{3,2\}}_{i_3,i_4} \epsilon_{i_5,i_6,i_7,i_8,i_9,i_{10}} G^{\{3,2\}}_{i_7,i_8} G^{\{3,2\}}_{i_9,i_{10}}. \tag{43}$$

– $\delta = 3$. The invariants are the $D^{(3)}$ discussed above.
– $\delta = 2$. The only invariant is $r$ defined above.
– $\delta = 1$. $G^{\{m,0\}}$
normalization forces the only invariant, the scalar $G^{\{m,0\}} = 1$, to be counted with $\delta = 1$. The term of order $\lambda^0$, the invariant $D^{(3)}$, should be read as $(D^{(3)} \cdot G^{\{3,0\}})$.

• *Visualization of 'tunnel' domains*. We now illustrate the domains for the matrix elements $G^{\{m,2\}}_{i,k}$ prescribed by the probability interpretation of the eigenvalues (30) (as a reminder, these formulae hold exactly for $m = 2$ and for $m \geqslant 3$ when we demand certain scalar products of pseudo-tensor(vector) configurations vanish, $D^{(3)}$ for $m = 3$). For obvious reasons we restrict the configurations to three non-vanishing matrix elements, e.g.

$$\begin{array}{lll} G^{\{m,2\}}_{1,2} := x & G^{\{m,2\}}_{3,4} := y & G^{\{m,2\}}_{2,3} := z \\ G^{\{m,2\}}_{i,k} := 0 & \text{otherwise.} & \end{array}$$

The eigenvalues then are

$$\lambda_{1,2} = \tfrac{1}{4}(1 \pm \alpha_+) \tag{44}$$

$$\lambda_{3,4} = \tfrac{1}{4}(1 \pm \alpha_-), \tag{45}$$

where

$$\alpha_\pm = \sqrt{(x \pm y)^2 + z^2}. \tag{46}$$

The domains are determined by the inequalities

$$0 \leqslant \alpha_+ \leqslant 1 \wedge 0 \leqslant \alpha_- \leqslant 1,$$

the admissible domains have to be subsets of these parameter regions which graphically represent two 'orthogonal' 'tunnels' with symmetry axes $y = \pm x$ and elliptic cross sections, half-axes 1 and $\sqrt{\frac{1}{2}}$ as depicted in figure 2. The physical domain is finally constructed as the intersection $\mathfrak{Int} = \text{tunnel}_{\alpha_+} \cap \text{tunnel}_{\alpha_-} \cap \{[x, y, z] | 0 \leqslant x, y, z \leqslant 1\}$ where the last set, the cube with edges $[x_0, y_0, z_0], x_0, y_0, z_0 = 0, 1$ represents the positivity condition, the correct normalization is guaranteed by (44) and (45). Figure 3 illustrates this intersection.

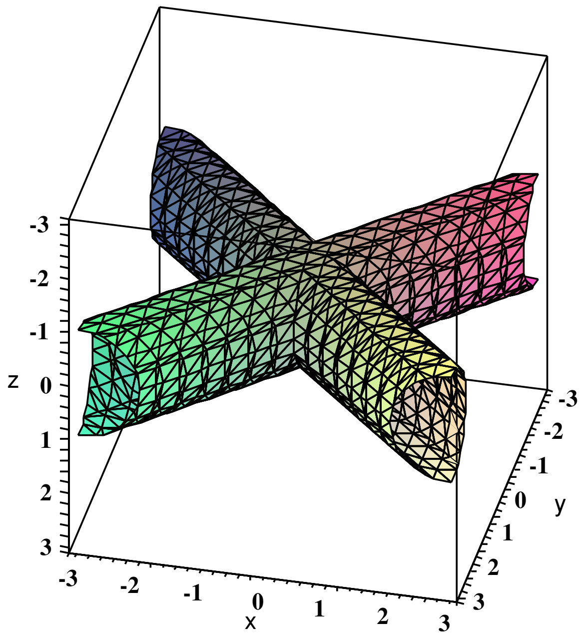


**Figure 2.** The domains (46) as a function of unrestricted $\{x, y, z\}$.

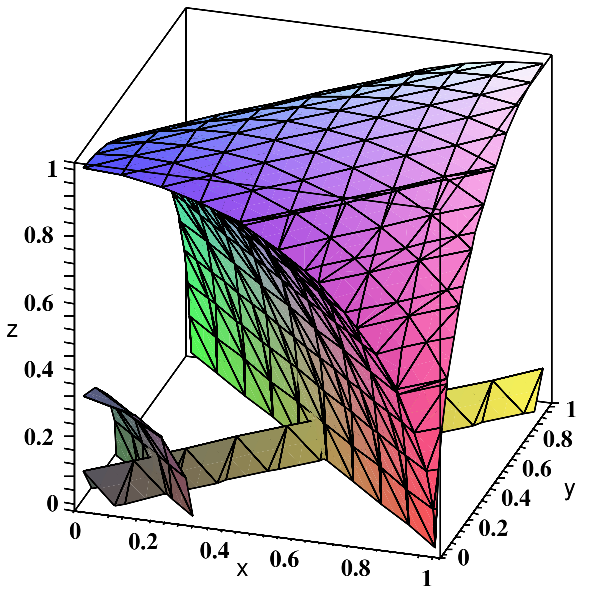


**Figure 3.** The intersection $\mathfrak{Int}$: surfaces $\alpha_+ = 1.0$ and $0.1$; $\alpha_- = 1.0$ and $0.01$ are depicted.

## 5. An alternative parameter classification

In the following I shall describe a classification which has the charm of accommodating a larger set of parameters into one tensor representation but is algebraically incomplete. Whether this has disadvantages when it comes to physical applications can be decided only after a clarification of the role of discrete transformations (similar to parity and charge conjugation in the Dirac theory). We postpone such questions and proceed as follows.

Put

$$\check{\Gamma}^{\{m,1\}} := \left[\hat{\Gamma}_1^{\{m,1\}}, \ldots, \hat{\Gamma}_{2m}^{\{m,1\}}, \Gamma_{2m+1}^{\{m\}}\right].$$

Then as already stated the $\check{\Gamma}_i^{\{m,1\}}$ fulfil the anti-commutation relations

$$\begin{aligned} &\check{\Gamma}_i^{\{m,1\}} \cdot \check{\Gamma}_j^{\{m,1\}} + \check{\Gamma}_j^{\{m,1\}} \cdot \check{\Gamma}_i^{\{m,1\}} = 2\delta_{i,j}\mathbb{I} \\ &i, j = 1, \ldots, 2m+1. \end{aligned} \tag{47}$$

Following equations (5) and (6) we write

$$\check{\Gamma}_{i_1,\ldots,i_k}^{\{m,k\}} = F_{\text{Norm}}^{\{m,k\}}\left(\varepsilon_{i_1,\ldots,i_{2m}} \check{\Gamma}_{i_{k+1}}^{\{m\}} \ldots \check{\Gamma}_{2m}^{\{m\}}\right).$$

The $2^m - 1$ parameters of an $m$-qubit are then accommodated in the expansion

$$\varrho^{m\text{-qubit}} = \sum_{k=0}^{m} \check{G}^{\{m,k\}} \circ \check{\Gamma}^{\{m,k\}},$$

where it is essential to note

$$\sum_{k=0}^{m} \binom{2m+1}{k} = 2^{2m}$$

the number of matrix elements defining the state $\varrho$.

The point to keep in mind is of course that this expansion is incomplete. However depending on the role of the already mentioned 'P'-, 'C'-transformations duality relations among the $\{m, k_0\}$ and $\{m, 2m - k_0\}$ tensors will resolve this problem.

This scheme takes care of a larger number of parameters

$m = 2.$

$$\text{Number of parameters} = \begin{cases} 5 & (4) \quad \text{for } k = 1 \quad \text{vector} \\ 5 + 10 & (4 + 6) \quad \text{for } k = 2 \quad \text{2-tensor.} \end{cases}$$

$m = 3.$

$$\text{Number of parameters} = \begin{cases} 7 & (6) \quad \text{for } k = 1 \quad \text{vector} \\ 7 + 21 & (6 + 15) \quad \text{for } k = 2 \quad \text{2-tensor.} \end{cases}$$

We now calculate the vector and 2-tensor spectra for this new representation:

For $k = 1$ the problem is already solved, see (37) and (38).

For $k = 2$ we have

$m = 2$. The formulae (30)

$$\lambda = \tfrac{1}{4}\left(1 \pm \sqrt{r \pm \sqrt{2r^2 - T_4}}\right),$$

as well as (31) and (32) hold with the replacement $G^{\{m,2\}} \longrightarrow \check{G}^{\{m,2\}}$.

$m = 3$. The same formulae hold if we replace the condition

$$D^{(3)} = 0 \tag{48}$$

by

$$\text{the } O(2m+1)\text{-invariant} \sum_{i=1}^{6} V_i^2 = 0 \tag{49}$$

$$\text{i.e. } V_i = 0 \quad i = 1, \ldots, 7$$

with

$$V_i = \epsilon_{i,i_1,\ldots,i_6} \check{G}^{\{3,2\}}_{i_1,i_2} \check{G}^{\{3,2\}}_{i_3,i_4} \check{G}^{\{3,2\}}_{i_5,i_6}. \tag{50}$$

Note that $V_7 = D^{(3)}$. For $m \geqslant 4$ the situation is a bit more involved. The corresponding maps (sub-'pseudovectors')[10]

$$\mathbb{M}(\mathbb{R}, 2m)_i \to \mathbb{R} \qquad i = 1, \ldots, 2m+1$$

of scale dimension $m$ constructed analogously to (50):

$$V_{i_1} = \epsilon_{i_1,\ldots,i_{2m+1}} \check{G}^{\{m,2\}}_{i_2,i_3} \ldots \check{G}^{\{m,2\}}_{i_{2m},i_{2m+1}}$$

carry too high dimension and play no role on the 2-tensor level. Suitable 'pseudo-tensors' have to be constructed and contracted to (pseudo-)scalars of the required dimension 6. The normalization of states is fixed once and for ever by normalizing the scalar term, positivity is guaranteed by the same inequalities among now $O(2m+1)$-invariants obtained above.

## 6. Summary

Given a Hermitian matrix with unit trace the decision whether or not it is a state is not at all trivial. More precisely speaking the *a priori* construction of a matrix representation of a state, a density matrix, is non-trivial. The classic way to determine the eigenvalues of this matrix as a function of its matrix elements, to solve the characteristic equation, is in general feasible (by radicals and algebraic operations) only for dimension $\leqslant 4$. For higher dimensions e.g. the Descartes' rule can be applied to derive admissible parameter domains. Doublet ($m = 1$), quartet ($m = 2$), and eventually octet ($m = 3$) structures in $m$-qubit spectra can be handled in this way with tolerable effort. Therefore a systematic study of generacy structures in $m$-qubit spectra seems essential.

The key of the approach we followed is to embed $m$-qubit states in Clifford algebras $\mathfrak{Cl}_{2m}$. The construction of a basis of this algebra from Clifford numbers obeying the anti-commutation rules (1) and (47) leads, considering the dual representation (9) of the algebra, to a classification of states as $O(2m)$- or $O(2m+1)$-tensors; the eigenvalues of these states are functions of $O(2m)$- and $O(2m+1)$-invariants. The number of parameters controlling positivity is thus considerably reduced. For $m$-qubits the case of degeneracy into doublets leads to a vector classification: state-parameters lie on $(2m-1)$- or $2m$-Bloch spheres. The degeneracy into quartets leads to more involved structures: relations among invariants and their embedding in parameter spaces are discussed in some detail. $m(2m \mp 1)$-dimensional intersections of 'tunnel'-like objects with elliptic cross-sections appear as generalizations of Bloch spheres. Progressing to tensors with $k_0 \geqslant 3$ one immediately encounters the obstacle of not explicitly knowing the eigenvalues as functions of $O(2m(+1))$-invariants, the already mentioned Descartes rule then comes into play.

The use of direct products of qubit states as basis in $m$-qubit state spaces has been proven useful for the development of criteria of separability, see e.g. [5–9]. Of particular interest in the present context are [2, 3] and references cited in these papers. There the basis $\{B_i\}$ is chosen as the generators of $SU(2^m)$, domains of admissible 'coherence vectors' guaranteeing positivity of the density matrix of an $m$-qubit are given in terms of Casimir invariants. In particular cases degeneracies were detected and the corresponding local unitaries described. Deriving explicit domains within this formalism soon encounters considerable problems (notwithstanding the essential structural clarifications gained): to determine the admissible domain for an $m$-qubit one obtains $2^m - 1$ polynomial inequalities with the maximal degree $2^m$:

[10] $\mathbb{M}(\mathbb{R}, 2m)$ is the space of $2m \times 2m$ real matrices.

writing the characteristic polynomial as $P(\lambda) = \sum_{i=1}^{2^m}(-1)^i a_i \lambda^i \; a_{2^m}$ is of scale dimension $2^m$ in the 'coherence vector' $\vec{n}$ (scale dimension $(a_i) = i$); the necessary and sufficient condition for positivity of the density matrix $a_i > 0$ for all i, $a_i = a_i(\vec{n})$.

The approach we follow leads to a $O(2m(+1))$-tensor classification and the corresponding degeneracy patterns. Domains of admissible parameters can thus be derived for *all* $m$ with increasing complexity for increasing order of the $O(2m(+1))$-tensors (the $k_{\text{tensor}} = 1, 2, 3$ cases can be comfortably handled with a standard laptop).

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